AN APPLICATION OF THE MARKOV PROCESS

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1. Markov process

Assume that we study evolution over time of a physical or ecological system and x(t) is position of the system at time t. The set of possible positions of the system is called state-space. Before the time s, the system is in a certain state and at s the system is in a state i. We want to know at the future time t, what is the probability that the system will be in the state j. If the probability depends on s, t, i and j, the evolution of the system in future only depends on the present and independent from the past. This is the Markov property. The process that expresses the evolution of a system with such a property is called the Markov process. We consider here a Markov process with a discrete set of times, or discrete-time Markov chain.

2. Transition probability

Consider (X_n); n = 0, 1, 2...as a homogeneous Markov chain, that is $p_{ij} = P(X_{n+1} = j/X_n=i) = P(X_{n+1} = j/X_0=i_0, ..., X_n=i)$ independent from n, where:

p_{ij}: conditional probability that the system in the state i at the time n goes to the state j at the time n+1.

Probability that the system in the state i at the starting time, after n steps, goes to the state j is determined by:

$$p_{ij}^{(n)} = P\left(X_{n+m} = j/X_m = i\right) = P(X_n = j/X_o = i)$$

and $P^{(n)} = \left(p_{ij}^{(n)}\right)$ is the transition probability matrix after n steps.

Distribution of the system at the time n is determined by

$p_j^{(n)} = P(X_n = j)$;n=0,1,2,...,j is the state

Let $B^{(n)} = \left(p_j^{(n)}, j\,state\right)$ and B = $B^{(0)}$ as initial distribution of the system. Then we have:

$$\mathbf{B^{(n)} = B.P^{(n)}}$$

Thus, model of a discrete and homogeneous Markov chain comprises three parties (X_n, B, P), where:

(X_n): set of discrete random quantities

B: initial distribution

P: transition probability matrix

3. Market share dividing model

Assume that N supermarkets sell the same product. Customers can buy it from one of N supermarkets. Choice of supermarket is up to customers and they can go to other ones for some reason. Supermarkets have to take various marketing measures to attract the customers.

This is a Markov chain containing N states (N supermarkets), transition probability pij is the probability that the customer moves from the supermarket i to j after some time.

The Markov chain model has:

+ State-space E = {1,2, ..., N}

+ Transition probability matrix: $P = (p_{ij})$ NxN

We use the Markov chain to:

- Predict number of customers in each time cycle for supermarkets.

- Predict market shares for supermarkets in future.

- Predict changes in the number of customers in future.

- Predict the future balance point.

Let's call x_1, x_2, x_3,..., x_N proportions of customers of supermarkets 1, 2, 3,..., N respectively. According to the marginal distribution, we look for non-negative solutions (x_1, x_2, ..., x_N) to set of equations:

$$\begin{cases} P^t . \begin{bmatrix} x_1 \\ x_2 \\ ... \\ x_N \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ ... \\ x_N \end{bmatrix} \\ x_1 + x_2 + ... + x_N = 1 \\ x_i \geq 0 + ; j = 1,2,...,N \end{cases}$$

where P^t is the transpose of the matrix P. If the initial distribution is B, we can work out the number of customers in the next time cycle by calculating: $B^{(1)} = B.P$.

Similarly, we can work out:

$B^{(2)} = B . P^2$; $B^{(3)} = B . P^3$; ... ; $B^{(n)} = B . P^n$

4. Use of the model to divide market shares between two airlines in Vietnam based on actual data

Call Vietnam Airline Company 1 and Pacific Airlines Company 2. They transport passengers between Hà Nội and HCMC. Each company must grasp changes in customers' preference if they want to carry out well their marketing campaigns. Thus, they should know how many passengers the two companies serve and work out effective measures to attract the passengers.

Consider $p_{jk}=P(X_{n+1}=k / X_n=j)$ (for all j, k) as the probability that the passenger flying j decides to fly k the next time; and $p_{ijk}=P(X_{n+1})=k / X_n=j,X_{n-1}=i)$ (for all i, j, k) as the probability that a passenger flies i in the first time, j in the second time and k in the third time.

If we have:

$p_{ijk} = p_{jk}$ (for all i, j, k) (*)

the process of changing from one company to another has Markov property.

To check whether the equation (*) is satisfactory or not, we use the following statistical standard: Calculate

$$T_{kij} = \frac{\left| \frac{n_{kij}}{n_{ki}} - \frac{n_{ij}}{n_i} \right|}{\sqrt{\overline{f}(1-\overline{f})(\frac{1}{n_{ki}} + \frac{1}{n_i})}} ;(k,i,j = 1,2)$$

where:

+ n_{ijk} (i, j, k = 1, 2): number of passengers who flied i in their first time, j in their second time and k in their third time.

+ n_{ij} (i, j = 1, 2): number of passengers who flied i in their first time, j in their second time.

ni (i = 1,2): number of passengers who flied i.

+ $\overline{f} = \frac{n_{kij} + n_{ij}}{n_{ki} + n_i}$

With significance of α, we work out t_α from the Laplace transform.

- If $T_{kij} \leq t_\alpha$, for all i, j, and k, we accept the hypothesis (*).

- If we have i, j, and k while $T_{kij} > t_\alpha$, we reject the hypothesis (*).

With 12,833 passengers who flied at least two times (data from the two companies) from HCMC to Hà Nội, by working out the transition probability matrix and checking the Markov property as mentioned above, we use a specialized software and find out:

N = 12.833 total number of passengers who flied at least two times.

n_1 = 10.460 : passengers who flied company 1.

n_2 = 2.373 : passengers who flied company 2.

n_{11} = 8.689 : passengers who moved from company 1 → 1.

n_{12} = 1.771 : passengers who moved from company 1 → 2.

N_{21} = 1.771 : passengers who moved from company 2 → 1.

N_{22} = 602 : passengers who moved from company 2 → 2.

We work out the transition probability matrix:

$$P = \begin{bmatrix} 0.8307 & 0.1693 \\ 0.7463 & 0.2537 \end{bmatrix}$$

Values of T_{kij} and comparing with $t_{\alpha} = 2.58$ (α= 1%).

We have: $T_{111} = 1.61452$; $T_{112} = 2.52116$; $T_{121} = 2.40033$; $T_{122} = 2.41365$

$T_{211} = 2.20114$; $T_{212} = 1.92754$; $T_{221} = 2.56420$; $T_{222} = 1.24162$

From this result, we see that passenger's decision to choose another company in their third flight is dependent on their second flight and independent from the first one. This means that this change approximates to the Markov process.

Let x_1 and x_2 be percentages of passengers of the two airlines. According to the model, they are solutions of the set of equations:

$$\begin{cases} p^t.\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ x_1 + x_2 = 1 \\ x_1, x_2 \geq 0 \end{cases} \text{ hay } \begin{cases} 0.1693x_1 - 0.7463x_2 = 0 \\ x_1 + x_2 = 1 \\ x_1, x_2 \geq 0 \end{cases}$$

Solving the set of equations, we get: x_1 = 0,8151; and x_2 = 0,1849

This means that in future, percentages of loyal passengers of the two companies are:

- Company 1: 81,51% of passengers.
- Company 2: 18,49% of passengers.

This result can help companies work out the most profitable business plans■

References

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