IN SEARCH OF THE LOWEST COST FOR A FIXED OUTPUT

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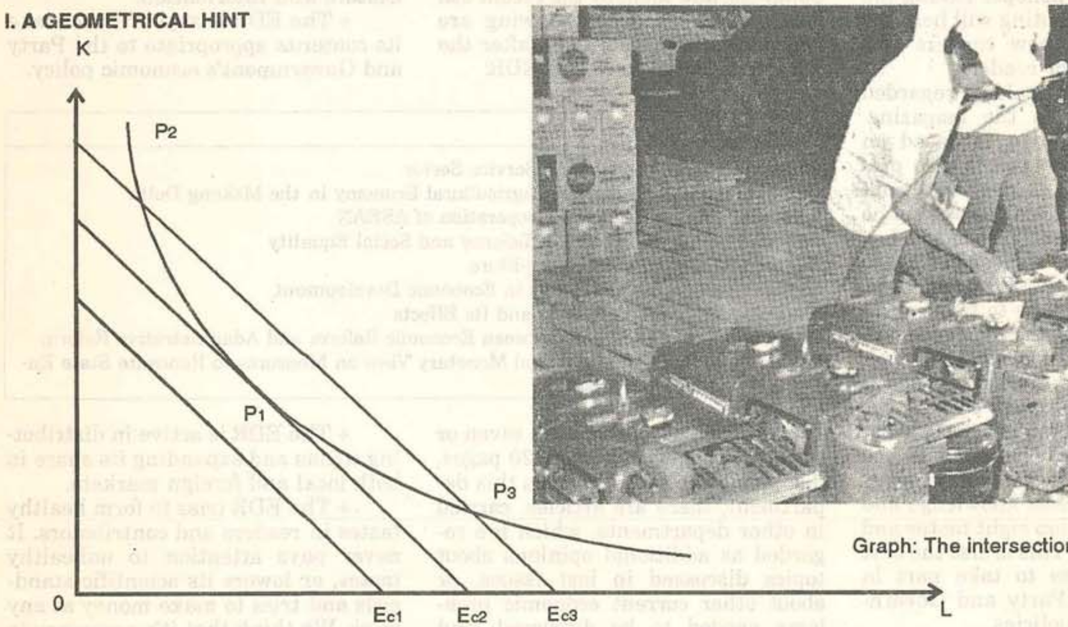
I. A GEOMETRICAL HINT



Graph: The intersection of E_p and E_c

We try putting an output curve E_p and three cost lines E_{c1}, E_{c2}, E_{c3} on the same coordinate axes OLK.

Suppose: The curve E_{c1} do not intersect E_p

The line E_{c2} is contiguous to E_p at point P_1

The line E_{c3} intersects E_p at P_2 and P_3

So if we express q as the corresponding output on E_p and TC_1, TC_2, TC_3 are total costs corresponding to three lines E_{c1}, E_{c2}, E_{c3}, we can have some following hints (With $TC_1 < TC_2 < TC_3$)

• The cost TC_1 cannot produce the output q, because E_{c1} do not intersect E_p.

• The production cost for output q at P_2 and P_3 is TC_3. This cost is not lowest.

• The cost for output q at P_1 is TC_2. This cost is the lowest one. Why? Because if there is a certain cost TC' smaller than TC_2, then TC' cannot produce output q because TC' does not meet Ep. If TC' is bigger than TC_2 then TC' is not the lowest cost for output q, because TC' intersects Ep at two points, the output at these two points is q, the cost is TC' > TC_2

• So the contiguous point between E_p and E_c is a position of coordination between L and K (point (L, K)) so that the production cost for output q is lowest, noted as TC_{min}

From the above hint, we will use the method of Lagrange multiplier to prove the existence and identify the adjacent point of E_p and E_c, thereby the lowest cost for a fixed output can be found.

** The extremes with accessive condition*

Let $Z = f(x_1, x_2, \dots, x_n)$ be the function of variables $x_1, x_2, \dots, x_n$.

We must determine the maxima, minima of f, when x_j, $j = \overline{1,n}$ must satisfy m conditions

$g_i(x_1, \dots, x_n) = 0$, $i = \overline{1,m}$

** The method of Lagrange multiplier*

The Lagrange function corresponding to the above problem is:

$$L(x_1, \dots x_n, \lambda_1, \dots, \lambda_m) = f(x_1 \dots x_n) + \sum_{i=1}^{m} \lambda_i \, g_i(x_i, \dots, x_n)$$

λ_i, $i = \overline{1,m}$ are called Lagrange multipliers.

The search of extremes of f with accessive conditions $g_i(x_1, \dots, x_n) = 0$ with $i = \overline{1,m}$ becomes the search of extremes without accessive condition of Lagrange function $L(x_1, \dots, x_n, \lambda_1, \dots, \lambda_m)$

THEOREM

The necessary conditions for function f to reach extremes at point $P(x_1^0, \dots, x_n^0)$ are:

$$\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} + \lambda i \frac{\partial g_i}{\partial x_j} = 0, \ j = \overline{1,n}$$

$$\frac{\partial L}{\partial \lambda_i} = g_i(x_1, \dots, x_n) = 0, \ i = \overline{1,m}$$

Application:

We assume $x_1, \dots, x_n$ are n inputs (n kinds of material, bi-products) used for producing an output with the production function

$q = q(x_1, x_2, \dots, x_n)$

Each bought kind of material corresponds to prices $k_1, \dots, k_n$. The enterprise has the function of total cost as follows:

$$TC = TC(x_1 ... x_n) = \sum_{i=1}^{n} k_i x_i$$

Problem: Find the volume of material of each kind which must be used for producing a fixed output $q = \overline{q}$, on the condition that the total cost of production is lowest.

• The problem in case of two variables

We examine the problem with two inputs x_1, x_2

Then we have:

The production function: $q = q(x_1,x_2)$

The total cost function: $TC = TC(x_1,x_2) = k_1x_1 + k_2x_2$

If $\overline{q}$ is a given output, then the equation of the output curve is $q = q(x_1,x_2) = \overline{q}$

Let TC have different values, the equation of the cost curve is:

$$x_2 = -\frac{k_1}{k_2} x_1 + \frac{TC}{k_2}$$

To find the production method for the lowest production cost and still secure the output q, we must solve the problem:

$TC(x_1,x_2) = k_1x_1 + k_2x_2 \rightarrow \min$

with $q(x_1,x_2) = \overline{q}$

The corresponding Lagrange function is:

$$L(x_1, x_2) = k_1x_1 + k_2x_2 + \lambda [\overline{q} - q(x_1,x_2)]$$

According to the above theorem, the necessary conditions for $P(x_1,x_2)$ to be the maximum point of $TC(x_1,x_2)$ with $q(x_1,x_2) = \overline{q}$ will be:

$$\frac{\partial L}{\partial x_1} = k_1 - \lambda \frac{\partial q}{\partial x_1} = 0$$

$$\frac{\partial L}{\partial x_2} = k_2 - \lambda \frac{\partial q}{\partial x_2} = 0$$

$$\frac{\partial L}{\partial \lambda} = \overline{q} - q(x_1,x_2) = 0$$

From above equations, deducing:

$$k_1 = \lambda \frac{\partial q}{\partial x_1}$$

$$k_2 = \lambda \frac{\partial q}{\partial x_2}$$

$$q(x_1, x_2) = \overline{q}$$

Or $$-\frac{k_1}{k_2} = -\frac{\frac{\partial q}{\partial x_1}}{\frac{\partial q}{\partial x_2}} \qquad (1)$$

But we had (see the equation of cost curves)

$$\frac{dx_2}{dx_1} = -\frac{k_1}{k_2}$$

thus deducing:

$$\frac{dx_2}{dx_1} = -\frac{\frac{\partial q}{\partial x_1}}{\frac{\partial q}{\partial x_2}} \qquad (2)$$

The derivative of the function $q(x_1,x_2)$ on the condition:

$$q(x_1,x_2) - \overline{q} = 0$$

We also have:

$$\frac{\partial q}{\partial x_1} dx_1 + \frac{\partial q}{\partial x_2} dx_2 = 0$$

Or $$\frac{dx_2}{dx_1} = -\frac{\frac{\partial q}{\partial x_1}}{\frac{\partial q}{\partial x_2}} \qquad (3)$$

The condition (1) shows us, along the output curve $q(x_1,x_2) = \overline{q}$, the enterprise will produce with the lowest production cost $P(x_1,x_2)$ where the slope of the straight line $x_2 = -\frac{k_1}{k_2} x_1 + \frac{TC_{min}}{k_2}$ and the slope of the output line are the same. In other words, the straight line $k_1x_1 + k_2x_2 = \underline{T}C_{min}$ must be contiguous to the output curve $q(x_1,x_2) = \overline{q}$. The coordinates of the contiguous point is the maximum production method we should find.

Result 1: in an enterprise having its production function as $q = q(x_1,x_2)$, total costs function as $TC = TC(x_1,x_2) = k_1x_1 + k_2x_2$ (k_i is the corresponding price of x_i, $i = 1,2$).

The enterprise will produce a fixed output $\overline{q}$ with the lowest cost at the contiguous point $p(x_1^0,x_2^0)$ of the output curve $q(x_1,x_2) = \overline{q}$ and the cost curve $TC = k_1x_1 + k_2x_2$

* THE RESULT EQUIVALENT TO RESULT 1

From (1) we deduce:

$$\frac{\frac{\partial q}{\partial x_1}}{k_1} = \frac{\frac{\partial q}{\partial x_2}}{k_2} \quad (4)$$

We know $\frac{\partial q}{\partial x_i}$, $i = 1,2$ is the marginal product corresponding to x_i, that is the additionally produced output when an unit of innput is added.

And: k_i is the cost to buy a unit of input x_i.

So we can express the economic meaning of (4) as follows:

The enterprise can select the production method with the lowest total costs, when the ratio between marginal product and the buying price of a unit of production input, between production factors is unchanged.

That means, if the input x_2 is more costly than x_1 ($k_2 > k_1$) then the enterprise will select the production method for $\frac{\partial q}{\partial x_1} < \frac{\partial q}{\partial x_2}$ (the more costly input will make the marginal product higher)■